Escape over a fluctuating potential barrier with three-state Markovian noise

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We consider the escape over a fluctuating potential barrier with a three-state Markovian noise. The resonant activations (RA's) for the mean first-passage times as functions of the transition rates γ_1, γ_2, and γ_3 of the three-state Markovian noise are obtained, respectively. The effect of the three values of the three-state Markovian noise on the RA's is investigated.

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I. INTRODUCTION

Recently the conventional problems of the escape over the fluctuating potential barrier have attracted a great deal of attention [1–20]. It was shown that the mean first-passage time (MFPT) of a particle driven by additive noises over a fluctuating potential barrier exhibits a minimum as a function of the flipping rate of the fluctuating potential barrier [1–20] (or the transition rate of the dichotomous noise). This phenomenon is called "resonant activation" and was first identified by Doering and Gadoua [1] and further studied by a number of other authors [2–20].

Earlier studies of activation of MFPT over fluctuating potentials were restricted to limiting cases—i.e., slow [21] or fast [21,22] barrier fluctuations, or small fluctuations [23]. Owing to using approximate treatments in Refs. [21–23], the resonant activation cannot be observed. Recently in Refs. [1–20], the authors reported results concerning the escape time (i.e., MFPT) over a fluctuating potential in the absence of approximate treatments as in Refs. [21–23]. They revealed the resonant activation (RA) of MFPT over the fluctuating potential barrier.

However, all of the above phenomena of resonant activation for the escape time are associated with the fluctuating potential barriers, which are dichotomous or Ornstein-Uhlenbeck. In this paper, we will consider the escape time over the fluctuating barrier, which is three-state Markovian.

II. MODEL AND ITS MASTER EQUATION

We consider a model whose Langevin equation is (in dimensionless form)

$$\dot{x} = -\partial_x U(x) + \xi(t) + \eta(t), \tag{1}$$

where $\xi(t)$ is a three-state Markovian noise and $\eta(t)$ is a Gaussian white noise. $U(x)$ is the potential. The fluctuating potential $U(x,t)$ satisfies

$$\partial_x U(x,t) = \partial_x U(x) - \xi(t). \tag{2}$$

Here $U(x)$ is piecewise linear (see Fig. 1). $\xi(t)$ takes values a, b, and c (a, b, and c are constants). The transition rates of $\xi(t)$ from a to b or vice versa, from b to c or vice versa, and from a to c or vice versa are, respectively, γ_1, γ_2, and γ_3. $\eta(t)$ has zero mean and correlation function $\langle \eta(t)\eta(t')\rangle = 2D\delta(t-t')$. Now, we assume that there is no correlation between $\eta(t)$ and $\xi(t)$.

The master equation for Eq. (1) is [24]

$$\partial_t P_1 = -\gamma_1 P_1 - \gamma_3 P_1 + \gamma_3 P_3 + \gamma_1 P_2 - \partial_x[-\partial_x U(x) + a]P_1 + D\partial_x^2 P_1,$$

$$\partial_t P_2 = -\gamma_1 P_2 - \gamma_2 P_2 + \gamma_1 P_1 + \gamma_2 P_3 - \partial_x[-\partial_x U(x) + b]P_2 + D\partial_x^2 P_2,$$

$$\partial_t P_3 = -\gamma_2 P_3 - \gamma_3 P_3 + \gamma_2 P_2 + \gamma_3 P_1 - \partial_x[-\partial_x U(x) + c]P_3 + D\partial_x^2 P_3, \tag{3}$$

where $P_1 = P(x,t,a)$, $P_2 = P(x,t,b)$, and $P_3 = P(x,t,c)$. $P_1 = P(x,t,a)$ represents that the particle is at x, the potential in $U(x)$, and the three-state Markovian noise in $\xi(t)=a$ configuration. There is the same understanding for P_2 and P_3.

We start with the potential at $x=-L/2$. So the initial condition is $P(x,0)=\sum_{i=1}^{3} P_i(x,0)=\delta(x+L/2)$. The boundary conditions for the reflecting ($x=-L/2$) and absorbing ($x=0$) boundaries, respectively, are $\{-[-\partial U(x)+a_i]P_i(x,t) + D\partial_x P_i(x,t)\}|_{x=-L/2}=0$ ($a_i=a,b$, and c) and $P_i(x,t)|_{x=0}=0$.

III. MEAN FIRST-PASSAGE TIME

The backward master equation for master equation (3) is [24]

$$\partial_t G_1 = -\gamma_1 G_1 - \gamma_3 G_1 + \gamma_3 G_3 + \gamma_1 G_2 + [-\partial_x U(x) + a]\partial_x G_1 + D\partial_x^2 G_1,$$

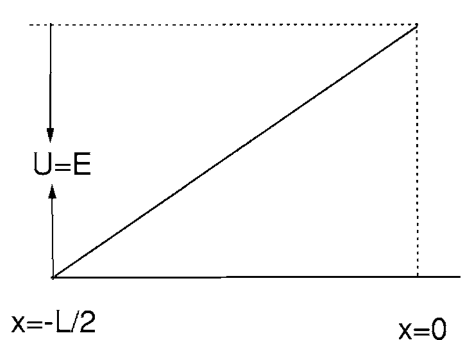

FIG. 1. The piecewise linear potential. For the numerical simulation in this paper, we take $L=2$.

$$\partial_t G_2 = -\gamma_1 G_2 - \gamma_2 G_2 + \gamma_1 G_1 + \gamma_2 G_3 + [-\partial_x U(x) + b]\partial_x G_2 + D\partial_x^2 G_2,$$

$$\partial_t G_3 = -\gamma_2 G_3 - \gamma_3 G_3 + \gamma_2 G_2 + \gamma_3 G_1 + [-\partial_x U(x) + c]\partial_x G_3 + D\partial_x^2 G_3. \quad (4)$$

The MTPT is defined as [24]

$$T_i(x) = -\int_0^\infty t\partial_t G_i(x,t)dt = \int_0^\infty G_i(x,t)dt, \quad (5)$$

where $i=1$, 2 and 3.

From Eqs. (4) and (5), one can obtain the equations of the MFPT:

$$-\gamma_1 T_1 - \gamma_3 T_1 + \gamma_3 T_3 + \gamma_1 T_2 + [-\partial_x U(x) + a]\partial_x T_1 + D\partial_x^2 T_1 + 1 = 0,$$

$$-\gamma_1 T_2 - \gamma_2 T_2 + \gamma_1 T_1 + \gamma_2 T_3 + [-\partial_x U(x) + b]\partial_x T_2 + D\partial_x^2 T_2 + 1 = 0,$$

$$-\gamma_2 T_3 - \gamma_3 T_3 + \gamma_2 T_2 + \gamma_3 T_1 + [-\partial_x U(x) + c]\partial_x T_3 + D\partial_x^2 T_3 + 1 = 0, \quad (6)$$

where T_i ($i=1$, 2, and 3) is the MFPT corresponding to the probability density P_i. The reflecting boundary condition is $\partial_x T_i(-L/2)=0$ and the absorbing boundary condition $T_i(0)=0$. The MFPT for a particle over the fluctuating barrier that starts at $x=-L/2$ is $T=\Sigma_{i=1}^3 T_i(-L/2)$.

Taking $\partial_x T_i = s_i$ $(i=1,2,3)$, Eq. (6) can be written as

$$\partial_x \begin{pmatrix} s_1 \\ T_1 \\ s_2 \\ T_2 \\ s_3 \\ T_3 \end{pmatrix} = \begin{pmatrix} (E-a)/D & (\gamma_1+\gamma_3)/D & 0 & -\gamma_1/D & 0 & -\gamma_3/D \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_1/D & (E-b)/D & (\gamma_1+\gamma_2)/D & 0 & -\gamma_2/D \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\gamma_3/D & 0 & -\gamma_2/D & (E-c)/D & (\gamma_2+\gamma_3)/D \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} s_1 \\ T_1 \\ s_2 \\ T_2 \\ s_3 \\ T_3 \end{pmatrix} + \begin{pmatrix} -1/D \\ 0 \\ -1/D \\ 0 \\ -1/D \\ 0 \end{pmatrix}. \quad (7)$$

When $3E-(a+b+c) \neq 0$, the solution of Eq. (7) is (see the Appendix)

$$s_i = \sum_{j=1}^5 K_j^{(i)} A_j^{(1)} \exp(\lambda_j x) + \frac{3}{3E-(a+b+c)}, \quad (8)$$

$$T_i = \sum_{j=1}^5 \frac{K_j^{(i)} A_j^{(1)}}{\lambda_j} \exp(\lambda_j x) + \frac{3x}{3E-(a+b+c)} + B_6^{(1)} + M_i, \quad (9)$$

where where $i=1,2,3$, and λ_j $(j=1,2,3,4,5)$ is the nonzero eigenvalues of the matrix of the homogeneous part about T_i and s_i $(i=1,2,3)$ in Eq. (7). Substituting Eqs. (8) and (9) into the boundary conditions $T_i(0)=0$ and $s_i(-L/2)=0$ $(i=1,2,3)$, we can obtain six linear algebraic equation for $A_j^{(1)}$ $(j=1,2,3,4,5)$ and $B_6^{(1)}$. From these equations, we can derive $A_j^{(1)}$ and $B_6^{(1)}$. Then, the MFPT for a particle over the fluctuating barrier reads

$$T = \sum_{i=1}^3 T_i(-L/2) = \sum_{i=1}^3 \sum_{j=1}^5 \frac{K_j^{(i)} A_j^{(1)}}{\lambda_j} \exp(-L\lambda_j/2) + 3B_6^{(1)} + M_2 + M_3 - \frac{3L/2}{3E-(a+b+c)}, \quad (10)$$

Here, the condition for the validity of Eq. (10) is $3E-(a+b+c) \neq 0$.

IV. CONCLUSION AND DISCUSSION

In Figs. 2(a)–2(c) we plot the behavior of the logarithm of the MFPT's with respect to the logarithm of the rates γ_1, γ_2, and γ_3. The figures show that there are RA's in the dynamics of the MFPT's with an increase of the transition rates γ_1, γ_2 and γ_3. A reason for these RA's happening here is given below. The resonance in Figs. 2(a)–2(c) occurs when the crossing takes place with the fluctuation potential barrier must likely in $U=\min(E-a,E-b,E-c)$ configuration (i.e., the "down" configuration). Now the MFPT has a minimum for the transition rates of the three-state Markovian noise on the order of the inverse of the time required to cross the potential with the fluctuation potential barrier in $U=\min(E-a,E-b,E-c)$ configuration. In Figs. 2(a)–2(c) we plot the corresponding points where the transition time equals the MFPT over the fluctuating potential barrier with the fluctuation potential barrier in $U=\min(E-a,E-b,E-c)$ configuration. It is clear that this accords with the above reason for the RAs happening in Figs. 2(a)–2(c). Further study shows that the MFPT of the particle over the fluctuating potential barrier can display a global minimum in (γ_1, γ_2, γ_3) space.

Below we consider the effect of the values a, b, and c of the three-state Markovian noise on the RA's. In Figs. 3(a)–3(c), for different values of the three-state Markovian noise we represent the logarithm of the MFPT as a function of the logarithm of the transition rate γ_1 [Fig. 3(a) corresponds to $a=-18$, -6, -4, -2, 0, 4, 8, 15, and 20 with D

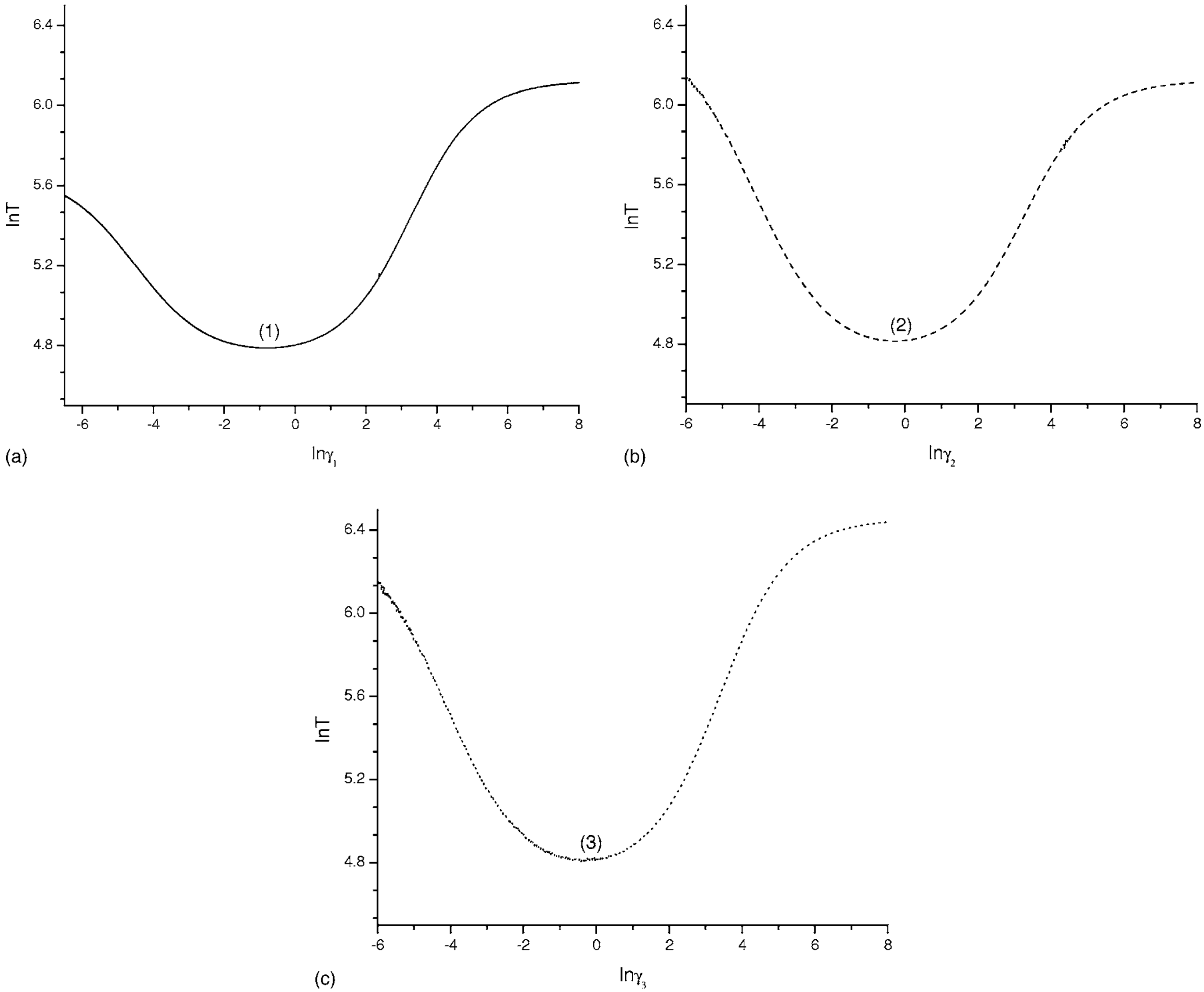

FIG. 2. (a) The logarithm of the MFPT versus the logarithm of the transition rate γ_1 of the three-state Markovian noise with $D=1$, $E=14$, $a=8$, $b=2$, $c=1$, $\gamma_2=\exp(-5)$, and $\gamma_3=\exp(-3)$; (b) the logarithm of the MFPT versus the logarithm of the transition rate γ_2 of the three-state Markovian noise with $D=1$, $E=14$, $a=1$, $b=8$, $c=2$, $\gamma_1=\exp(-5)$, and $\gamma_3=\exp(-3)$; (c) the logarithm of the MFPT versus the logarithm of the transition rate γ_3 of the three-state Markovian noise with $D=1$, $E=14$, $a=1$, $b=2, c=8$, $\gamma_2=\exp(-5)$, and $\gamma_1=\exp(-3)$. The marked points (1), (2), and (3) are the corresponding points where the transition time equals the MFPT over the fluctuating barrier with the fluctuating potential barrier in the "down" configuration.

$=1$, $E=14$, $b=2$, $c=1$, $\gamma_2=\exp(-5)$ and $\gamma_3=\exp(-3)$, Fig. 3(b) to $b=-16$, -10, -6, -4, -2, 0, 2, 6, 10, and 15 with $D=1$, $E=14$, $a=8$, $c=1$, $\gamma_2=\exp(-5)$, and $\gamma_3=\exp(-3)$, Fig. 3(c) to $c=-15$, -10, -6, 0, 6, 10, and 15 with $D=1$, $E=14$, $a=8$, $b=2$, $\gamma_2=\exp(-5)$, and $\gamma_3=\exp(-3)$]. From these figures we can find that (i) for the value a of the three-state Markovian noise, when a is zero or negative the RA is indistinct; with the increase of a, when a is positive, if it is smaller the resonant behavior is still indistinct; with the further increase of a, the RA becomes distinct; afterwards, the resonant behavior becomes indistinct again; (ii) for the value b of the three-state Markovian noise, when b is negative or zero the RA is distinct; while when b is positive, if b is smaller the resonant behavior is still distinct, if b is larger the resonance becomes indistinct; (iii) for the value c of the three-state Markovian noise, when c is negative or zero the resonant activation is distinct; while when c is positive, with the increase of c, the resonant behavior becomes more and more indistinct. Above, we consider the effect of a, b, and c on the RA for the MFPT as a function of γ_1. For the RA of the MFPT as a function of γ_2, study shows that if we replace a with b, γ_1 with γ_2, b with c, γ_2 with γ_3, c with a, and γ_3 with γ_1 of Figs. 3(a)–3(c), we can get the same figures as Figs. 3(a)–3(c) for the MFPT versus γ_2. So the effect of b on the RA of the MFPT as a function of γ_2 is same as a on the RA for the MFPT as a function of γ_1, the effect of a on the RA of the MFPT as a function of γ_2 is same as c on the RA for the MFPT as a function of γ_1, and c on the RA for the MFPT versus γ_2 is same as b on the RA for the MFPT as a function of γ_1. Similarly, further studying shows that for the

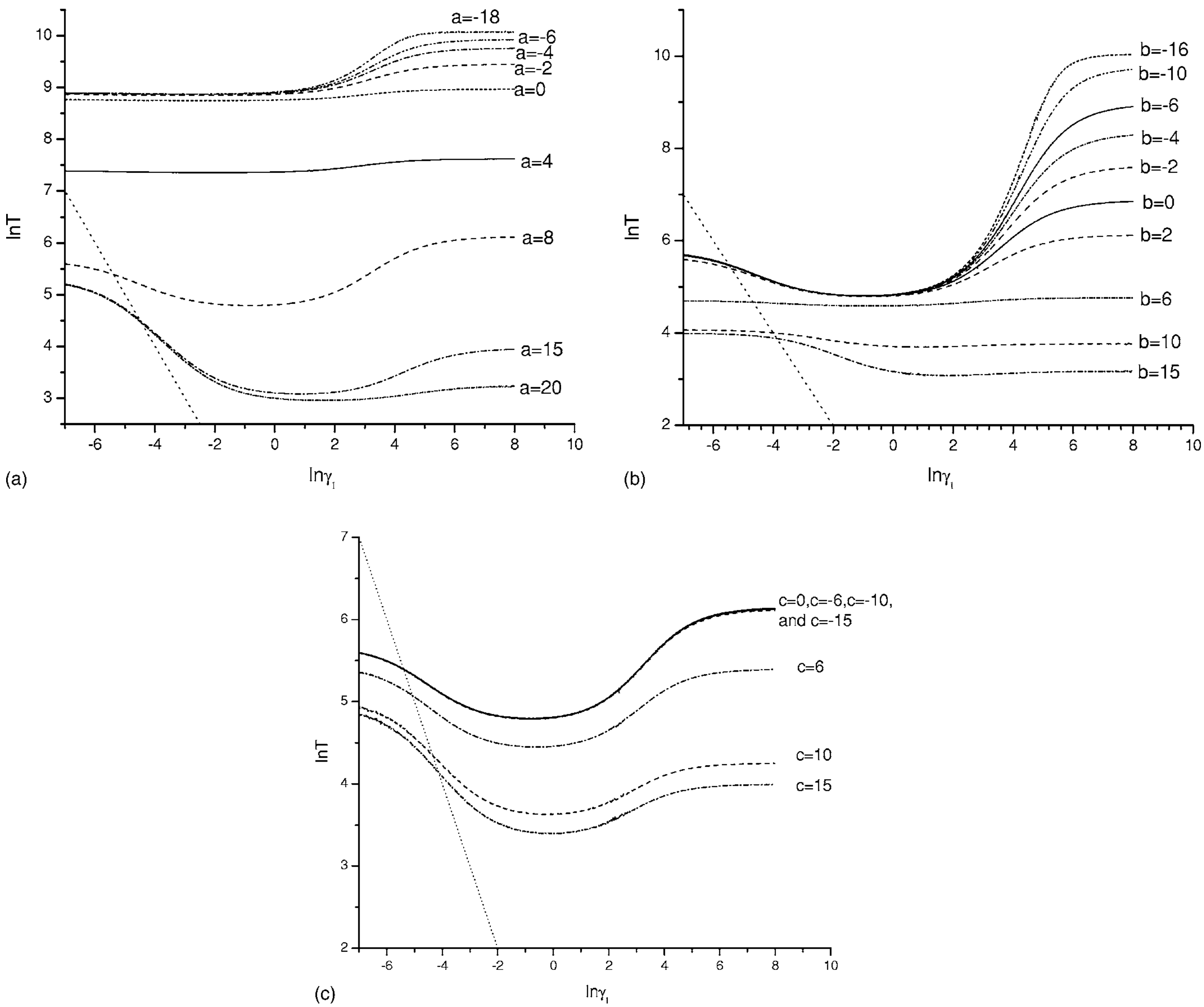

FIG. 3. The logarithm of the MFPT versus the logarithm of the transition rate γ_1 of the three-state Markovian noise with $D=1$, $E=14$, $\gamma_2=\exp(-5)$, and $\gamma_3=\exp(-3)$ (a) for different values of a ($a=-18$, -6, -4, -2, 0, 4, 8, 15, and 20, respectively) with $b=2$ and $c=1$, (b) for different values of b ($b=-16$, -10, -6, -4, -2, 0, 2, 6, 10, and 15, respectively) with $a=8$ and $c=1$, (c) for different values of c ($c=-15$, -10, -6, 0, 6, 10, and 15, respectively) with $a=8$ and $b=2$. The dotted lines are plotted for the case when γ' (i.e., $1/T$) equals γ_1.

RA of the MFPT as a function of γ_3, the effect of c on the RA of the MFPT as a function of γ_3 is same as a on the RA for the MFPT as a function of γ_1, the effect of a on the RA of the MFPT as a function of γ_3 is same as b on the RA for the MFPT as a function of γ_1, and b on the RA of the MFPT as a function of γ_3 is same as c on the RA for the MFPT as a function of γ_1.

The relation of our work to the phenomenon of stochastic resonance should be considered. For the phenomenon of stochastic resonance, we know that the response of a nonlinear stochastic system to an inputting signal will be enhanced by the presence of noise and maximized for certain value of the noise's strength. When the frequency of the inputing signal is equal to the intrinsic frequency of the original stochastic system, a phenomenon of resonance will appear. In our paper, the RA has the phenomenon of resonance. We analyze the phenomenon of resonance appearing in the RA below. For small values γ_1 (γ_2 or γ_3) of the transition rate of the fluctuating potential barrier, a destructive influence on the asymmetry of the system will be played, so the $\ln(1/T)$-$\ln\gamma_1$ $[\ln(1/T)$-$\ln\gamma_2$ or $\ln(1/T)$-$\ln\gamma_3]$ response curve will have positive slope. For large γ_1 (γ_2 or γ_3), a central role will be played in producing coherent motion with increases as γ_1 (γ_2 or γ_3) increases; then, the $\ln(1/T)$-$\ln\gamma_1$ $[\ln(1/T)$-$\ln\gamma_2$ or $\ln(1/T)$-$\ln\gamma_3]$ curve goes down. Thus finally we can obtain a peaked $\ln(1/T)$-$\ln\gamma_1$ $[\ln(1/T)$-$\ln\gamma_2$ or $\ln(1/T)$-$\ln\gamma_3]$ curve, at the peak of which a phenomenon of resonance appears. The intrinsic frequency of the stochastic system studied by us is $\gamma'=1/T$ in which T is the MFPT over the fluctuating potential barrier. When γ' is equal to the frequency of the fluctuating potential barrier—i.e., the transition rate γ_1 (γ_2 or γ_3)—no resonance happens. In Fig. 3(a), we plot the line when γ' is equal to γ_1 (the dotted line); in Fig. 3(b), the line when γ' equals γ_2 (the dotted line) is plotted; in Fig. 3(c), the line when γ' is equal to γ_3 (the dotted line) is plotted.

Below we consider the case when $3E-(a+b+c)=0$. Numerical simulation and analysis show that now there are four nonzero real independent eigenvalues and two zero eigenvalues for the matrix of the homogeneous part (note $\partial_x T_i = s_i$). So we have

$$s_i = \sum_{j=1}^{4} C_j^{(i)} \exp(r_j x) + C_5^{(i)} + C_6^{(i)} x,$$

$$T_i = \sum_{j=1}^{4} \frac{C_j^{(i)}}{r_j} \exp(r_j x) + D_5^{(i)} + C_5^{(i)} x + \frac{1}{2} C_6^{(i)} x^2, \tag{11}$$

in which r_j $(j=1,2,3,4)$ are the four nonzero eigenvalues of the matrix of the homogeneous part in Eq. (7). From Eq. (7) and the boundary conditions for T_i and s_i, using the similar method used above by us, we can get $C_j^{(i)}$ and $D_5^{(i)}$ $(i=1,2,3$, and $j=1,2,3,4,5,6)$. The MFPT for a particle over the fluctuating barrier is

$$T = \sum_{i=1}^{3} \left[\sum_{j=1}^{4} \frac{C_j^{(i)}}{r_j} \exp(-L\lambda_j/2) + D_5^{(i)} - \frac{L}{2} C_5^{(i)} + \frac{L^2}{8} C_6^{(i)} \right]. \tag{12}$$

Further study shows that when $3E-(a+b+c)=0$ there is the same phenomenon as reported above.

In Eq. (1), the noise is only three state noise. When it is four or more states, there will be the same phenomenon (i.e., there are the RA's for the MFPT's as the functions of the every transition rate of the noise). In addition, when the potential barrier is not piecewise linear, such as a bistable or multitable potential, as long as there is the three-state Markovian noise in Eq. (1) the phenomenon reported by us in the paper will exist.

Finally, it must be mentioned that the first consideration of exit-time statistics for multivalue Markovian noise was made by Hagan, Doering, and Levermore in Ref. [25], but they did not find the phenomenon of resonant activation; the three-state model considered by us in this paper is the same as consideration in the distinct context of ratchet transport by Elston and Doering [26], but they did not investigate the escape for the particle over the fluctuating potential barrier.

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APPENDIX

By numerical simulation and analysis we can find that when $3E-(a+b+c)\neq 0$ the matrix of the homogeneous part in Eq. (7) has five nonzero real independent eigenvalues and a zero eigenvalue. So the general solution of Eq. (7) is

$$s_i = \sum_{j=1}^{5} A_j^{(i)} \exp(\lambda_j x) + A_6^{(i)} + A_7^{(i)} x, \tag{A1}$$

$$T_i = \sum_{j=1}^{5} B_j^{(i)} \exp(\lambda_j x) + B_6^{(i)} + B_7^{(i)} x, \tag{A2}$$

where $i=1,2,3$, and λ_j $(j=1,2,3,4,5)$ is the nonzero eigenvalues.

Substituting Eqs. (A1) and (A2) into Eq. (7) and using the comparing coefficient method, we can obtain

$$A_6^{(1)} = A_6^{(2)} = A_6^{(3)} = \frac{3}{3E-(a+b+c)},$$

$$B_j^{(i)} = \frac{A_j^{(i)}}{\lambda_j}, \quad A_7^{(i)} = 0, \quad B_7^{(i)} = A_6^{(i)},$$

$$B_6^{(i)} = B_6^{(1)} + M_i, \quad \text{and } A_j^{(i)} = K_j^{(i)} A_j^{(1)},$$

with

$$M_1 = 0,$$

$$M_2 = \frac{(b+c-2a)\gamma_2 + (2b-a-c)\gamma_3}{(\gamma_1\gamma_2+\gamma_1\gamma_3+\gamma_2\gamma_3)[3E-(a+b+c)]},$$

$$M_3 = \frac{(b+c-2a)\gamma_2 + (2c-a-b)\gamma_1}{(\gamma_1\gamma_2+\gamma_1\gamma_3+\gamma_2\gamma_3)[3E-(a+b+c)]},$$

$$K_j^{(1)} = 1,$$

$$K_j^{(2)} = \frac{\gamma_1\gamma_2+\gamma_1\gamma_3+\gamma_2\gamma_3+\lambda_j\gamma_2(E-a) - D\gamma_2\lambda_j^2}{\gamma_1\gamma_2+\gamma_1\gamma_3+\gamma_2\gamma_3+\lambda_j\gamma_3(E-b) - D\gamma_3\lambda_j^2},$$

$$\text{and } K_j^{(3)} = -\frac{\gamma_1}{\gamma_3} K_j^{(2)} + \frac{\gamma_1+\gamma_3}{\gamma_3} + \frac{\lambda_j(E-a)}{\gamma_3} - \frac{D\lambda_j^2}{\gamma_3}.$$

Then, Eqs. (A1) and (A2) can be written as

$$s_i = \sum_{j=1}^{5} K_j^{(i)} A_j^{(1)} \exp(\lambda_j x) + \frac{3}{3E-(a+b+c)},$$

$$T_i = \sum_{j=1}^{5} \frac{K_j^{(i)} A_j^{(1)}}{\lambda_j} \exp(\lambda_j x) + \frac{3x}{3E-(a+b+c)} + B_6^{(1)} + M_i.$$

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